



Calhoun: The NPS Institutional Archive
DSpace Repository

Theses and Dissertations

1. Thesis and Dissertation Collection, all items

1971

A test of two unitary exponential samples with and without time truncation.

Swan, James Ned.

Monterey, California ; Naval Postgraduate School

<http://hdl.handle.net/10945/15941>

Downloaded from NPS Archive: Calhoun



Calhoun is the Naval Postgraduate School's public access digital repository for research materials and institutional publications created by the NPS community. Calhoun is named for Professor of Mathematics Guy K. Calhoun, NPS's first appointed -- and published -- scholarly author.

Dudley Knox Library / Naval Postgraduate School
411 Dyer Road / 1 University Circle
Monterey, California USA 93943

<http://www.nps.edu/library>

**A TEST OF TWO UNITARY
EXPONENTIAL SAMPLES WITH AND
WITHOUT TIME TRUNCATION**

JAMES NED SWAN

LIBRARY
NAVAL POSTGRADUATE
MORTON, CALIF. 97

United States Naval Postgraduate School



THESIS

A Test of Two Unitary Exponential Samples
With and Without Time Truncation

by

James Ned Swan

Thesis Advisor:

Toke Jayachandran

September 1971

Approved for public release; distribution unlimited.

A Test of Two Unitary Exponential Samples
With and Without Time Truncation

by

James Ned Swan
Lieutenant, United States Navy
B.S., United States Naval Academy, 1964

Submitted in partial fulfillment of the
requirements for the degree of

MASTER OF SCIENCE
(with major in mathematics)

from the
NAVAL POSTGRADUATE SCHOOL
September 1971

ABSTRACT

A test of the equality of the parameters λ_1 and λ_2 of two exponential random variables X_1 and X_2 is developed and evaluated for the case of a single sample from each population. The test is valid with or without time truncated testing although the test is less powerful under time truncated testing. The results are applicable to reliability studies.

TABLE OF CONTENTS

I.	INTRODUCTION -----	6
	A. PURPOSE -----	6
	B. BACKGROUND -----	6
	C. APPLICATIONS -----	8
II.	DERIVATION OF PROCEDURE -----	9
III.	ESTIMATION OF $\lambda_2 - \lambda_1$ -----	19
	A. POINT ESTIMATION -----	19
	B. CONFIDENCE INTERVALS -----	20
	C. APPLICATIONS OF CONFIDENCE INTERVALS -----	21
IV.	APRIORI KNOWLEDGE OF λ -----	22
	A. CHOICE OF TEST TIME T -----	22
V.	AREAS FOR FURTHER STUDY -----	23
	A. EXTENSION OF RESULT TO SAMPLES GREATER THAN ONE -----	23
VI.	CONCLUSIONS -----	26

LIST OF TABLES

I. SELECTED VALUES OF THE PROBABILITY OF TYPE II ERROR -----	16
--	----

LIST OF DRAWINGS

1.	POWER FUNCTION LOWER BOUND WITH TIME TRUNCATION -----	17
2.	POWER FUNCTION WITHOUT TIME TRUNCATION -----	18

I. INTRODUCTION

A. PURPOSE

The purpose of this thesis is to present a method for testing the equality of the scale parameters, λ_1 and λ_2 , of two exponential random variables X_1 and X_2 whose respective density functions are

$$\begin{aligned} \exp(-\lambda_1 x_1) & \quad \lambda_1 > 0, x_1 > 0 \\ \exp(-\lambda_2 x_2) & \quad \lambda_2 > 0, x_2 > 0 \end{aligned}$$

when time truncated testing of only one sample from each population is used. Time truncation consists of defining the new random variable Y where

$$Y_i = \begin{cases} X_i & \text{if } X_i < T \\ T & \text{if } X_i \geq T \end{cases} \quad i=1,2$$

B. BACKGROUND

Lentner and Buehler [Ref.1] investigated the problem of making inferences about the relative magnitudes of the scale parameters of two untruncated exponential random variables X_1 and X_2 when sampling from each population was of size greater than one. In this case the sum of the random samples from each population is considered to be the observation of a random variable Y_i whose distribution is gamma with parameters λ_i and n_i . The conditional distribution of the difference, $Y_1 - Y_2$ given the sum $Y_1 + Y_2$ is dependent on the parameter $\lambda_1 - \lambda_2$. This result holds for any distribution from the exponential family, that is, a distribution of the form

$$f_Y(y) = c(\lambda)h(y) \exp\left[\sum T_i(y) Q_i(\lambda)\right] .$$

This result was the basis for obtaining a confidence interval for the parameter $\lambda_1 - \lambda_2$.

Lloyd and Lipow [Ref. 2] present methods for obtaining confidence intervals for the parameter λ in the case of a single population Y with an exponential distribution. These procedures utilize the chi-square distribution of the difference $Y_{i+1} - Y_i$ $i=1, \dots, n$ of the order statistics, $0=Y_0 < Y_1 \leq Y_2 \leq Y_3 \leq \dots \leq Y_{n+1}$ for observations from an exponential distribution.

Cox [Ref. 3] uses the ranked observations from two exponential distributions and presents a modified F test for testing the hypothesis that both samples are from the same distribution, that is $\lambda_1 = \lambda_2$. The power, P [the statistic falls in rejection region], of this test was investigated by Gehan and Thomas [Ref. 4] for the case of time truncated testing and small sample sizes using computer simulation. The test statistic here is t_1/t_2 where

$$t_i = \frac{(v_i T + \sum_j y_{ij})}{(n_i - v_i)} = \frac{(v_i T + \text{sum of observed values from sample } i)}{(\text{number of observed values from sample } i)}$$

v_i = number of truncated observations

The test statistic has an approximate F distribution with degree of freedom $2(n_1 - v_1)$ and $2(n_2 - v_2)$. Their conclusion was that Cox's modified F test is nearly as powerful as the standard F test. The modified F test is not appropriate for the case of single samples since time truncation yields a zero divisor.

C. APPLICATIONS

The results obtained herein are applicable to reliability studies of components assuming that the times to failure are distributed as exponential random variables. The problem consists of choosing an alternate type of component to replace the type presently in use. In this case, high cost or large size might dictate the need to select an alternative with equal reliability when the items to be tested are in small supply and the time allocated is not sufficient for full testing. Another application for this procedure is the problem of making inferences when only one trial from each population was made, and outside considerations forced the termination of testing, for example, an electrical power failure during the initial testing of two limited production items.

II. DERIVATION OF PROCEDURE

A. DISTRIBUTION OF THE SUM AND DIFFERENCE UNDER TIME TRUNCATION

The random variable X with density function

$$\exp(-\lambda x) \quad \lambda > 0, x > 0$$

is transformed to the random variable Y with density function

$$\exp(-\lambda y) \quad 0 < y < T$$

$$\exp(-\lambda T) \quad y = T$$

when time truncated at time T . Using the simultaneous transformation

$$Z_1 = Y_1 + Y_2$$

$$Z_2 = Y_1 - Y_2$$

The joint density function of Z_1 and Z_2 may be found by use of Laplace transformations. First the density function of Y_i , $i=1, 2$, is rewritten

$$u(t) \left[\lambda \exp(-\lambda t) \right] - u(t-T) \left[\lambda \exp(-\lambda t) \right] \\ + u'(t-T) \left[\exp(-\lambda T) \right],$$

where $u(t-a)$ is the unit step function at $t=a$,

$$u(t-a) = 0 \quad t < a \\ = 1 \quad t \geq a$$

and $u'(t-a)$ is a unit mass function or Dirac delta function at $t=a$.

$u'(t-a)$ has the property

$$\int_{-\infty}^x u'(t-a) dt = 0 \quad x < a \\ = 1 \quad x \geq a$$

Now $\widetilde{f}_{Z_1 Z_2}(s_1, s_2) = E \left[\exp(-s_1 Z_1 - s_2 Z_2) \right]$ by definition and by independence of Y_1 and Y_2 this becomes

$$\widetilde{f}_{Z_1 Z_2}(s_1, s_2) = \widetilde{f}_{Y_1}(s_1 + s_2) \widetilde{f}_{Y_2}(s_1 - s_2) .$$

For this case, using standard tables of Laplace transformations,

$$\widetilde{f}_{Y_i}(s) = \left[\lambda_i / (\lambda_i + s) \right] \left[1 - \exp(-(\lambda_i + s)T) \right] + \exp \left[-(\lambda_i + s)T \right]$$

$$i = 1, 2.$$

By multiplication of $\widetilde{f}_{Y_1}(s_1 + s_2)$ and $\widetilde{f}_{Y_2}(s_1 - s_2)$, then inverting the transformation, first with respect to s_1 and then with respect to s_2 , the result, in terms of unit step functions and unit mass functions, is the joint density function of Z_1 and Z_2 . This density is

$$(1.1) \quad (\lambda_1 \lambda_2 / 2) \exp \left[-(\lambda_1 + \lambda_2)z_1 / 2 + (\lambda_2 - \lambda_1)z_2 / 2 \right] \quad \begin{array}{l} 0 < z_1 < T \\ |z_2| < z_1 \\ T \leq z_1 < 2T \\ |z_2| < 2T - z_1 \end{array}$$

$$(1.2) \quad \lambda_2 \exp \left[(\lambda_2 - \lambda_1)T - \lambda_2 z_1 \right] \quad \begin{array}{l} T \leq z_1 < 2T \\ z_2 = 2T - z_1 \end{array}$$

$$(1.3) \quad \lambda_1 \exp \left[(\lambda_1 - \lambda_2)T - \lambda_1 z_1 \right] \quad \begin{array}{l} T \leq z_1 < 2T \\ z_2 = z_1 - 2T \end{array}$$

$$(1.4) \quad \exp \left[-(\lambda_1 + \lambda_2)T \right] \quad \begin{array}{l} z_1 = 2T, z_2 = 0 \end{array}$$

when $\lambda_1 = \lambda_2 = \lambda$, this becomes

$$(1.1.1) \quad (\lambda^2/2) \exp[-\lambda z_1] \quad \begin{array}{l} 0 < z_1 < T, \quad |z_2| < z_1 \\ T \leq z_1 < 2T, \quad |z_2| < 2T - z_1 \end{array}$$

$$(1.2.1.) \quad \exp[-\lambda z_1] \quad T \leq z_1 < 2T, \quad z_2 = 2T - z_1$$

$$(1.3.1.) \quad \exp[-z_1 \lambda] \quad T \leq z_1 < 2T, \quad z_2 = z_1 - 2T$$

$$(1.4.1.) \quad \exp[-2\lambda T] \quad z_1 = 2T, \quad z_2 = 0$$

To obtain the conditional density of Z_2 given Z_1 , the marginal density of Z_1 is obtained by integration of the joint density with respect to Z_2 .

For example when $T \leq z_1 < 2T$, $f_{Z_1}(z_1)$ is

$$\int_{z_1 - 2T}^{2T - z_1} (\lambda_1 \lambda_2 / 2) \exp[-(\lambda_1 + \lambda_2)z_1/2 + (\lambda_2 - \lambda_1)z_2/2] dz_2 \\ + \lambda_2 \exp[(\lambda_2 - \lambda_1)T - \lambda_2 z_1] + \lambda_1 \exp[(\lambda_1 - \lambda_2)T - \lambda_1 z_1].$$

the marginal density of Z_1 is then

$$(1.5) \quad [(\lambda_1 \lambda_2) / (\lambda_2 - \lambda_1)] [\exp(-\lambda_1 z_1) - \exp(-\lambda_2 z_1)] \quad 0 < z_1 < T$$

$$(1.6) \quad [(\lambda_1 \lambda_2) / (\lambda_2 - \lambda_1)] [\exp(-\lambda_2 z_1 + \lambda_2 T - \lambda_1 T) \\ - \exp(-\lambda_1 z_1 - \lambda_2 T + \lambda_1 T)] + \lambda_1 \exp[-(\lambda_2 - \lambda_1)T - \lambda_1 z_1] \\ + \lambda_2 \exp[-\lambda_2 z_1 - (\lambda_1 - \lambda_2)T] \quad T \leq z_1 < 2T$$

$$(1.7) \quad \exp[-(\lambda_1 + \lambda_2)T] \quad z_1 = 2T$$

and when $\lambda_1 = \lambda_2 = \lambda$, this reduces to

$$(1.5.1) \quad (\lambda^2/2) [\exp(-\lambda z_1)] (2z_1) \quad 0 < z_1 < T$$

$$(1.6.1) \quad (\lambda^2/2) [\exp(-\lambda z_1)] 2(2T - z_1) \\ + 2\lambda \exp(-\lambda z_1) \quad T \leq z_1 < 2T$$

$$(1.7.1) \quad \exp(-2\lambda T) \quad z_1 = 2T.$$

Finally, the density of Z_2 given Z_1 is obtained by division of (1.1) by (1.5),

$$(1.8) \quad \left(\frac{\lambda_2 - \lambda_1}{2} \right) \left(\frac{\exp \left[(\lambda_2 - \lambda_1) (z_2 - z_1) / 2 \right]}{1 - \exp \left[-(\lambda_2 - \lambda_1) z_1 \right]} \right) \quad \begin{matrix} |z_2| < z_1 \\ 0 < z_1 < T \end{matrix}$$

and dividing (1.1) by (1.6)

$$(1.9) \quad \left(\frac{\lambda_2 - \lambda_1}{2} \right) \left(\frac{\exp \left[\left(\frac{\lambda_2 - \lambda_1}{2} \right) z_2 - \left(\frac{\lambda_1 + \lambda_2}{2} \right) z_1 \right]}{\lambda_2^2 \exp \left[-\lambda_2 z_1 - (\lambda_1 - \lambda_2) T \right] - \lambda_1^2 \exp \left[-\lambda_1 z_1 - (\lambda_2 - \lambda_1) T \right]} \right) \quad \begin{matrix} |z_2| < 2T - z_1 \\ T \leq z_1 < 2T \end{matrix}$$

The other portions of $f_{Z_2|Z_1}$ will not be used in the development and are not listed. When $\lambda_1 = \lambda_2 = \lambda$, the conditional density $f_{Z_2|Z_1}$ is

$$(1.8.1) \quad \begin{matrix} 1/2z_1 & |z_2| < z_1, & 0 < z_1 < T \\ 1/[2(2T - z_1) + 4/\lambda] & |z_2| < 2T - z_1, & T \leq z_1 < 2T \end{matrix}$$

A heuristic ratio test, similar to a likelihood ratio test, of $\lambda_1 = \lambda_2$ versus $\lambda_1 \neq \lambda_2$ using the density function f_{Z_1, Z_2} is a test which calls for rejecting the null hypothesis $H_0: \lambda_1 = \lambda_2$ when R defined as

$$\frac{f_{Z_1, Z_2}(z_1, z_2 | \lambda_1 = \lambda_2)}{f_{Z_1, Z_2}(z_1, z_2 | \lambda_1 \neq \lambda_2)}$$

is less than or equal to a fixed number K and calls for accepting H_0 when R is greater than K .

In the problem being considered, if such a K exists then solving

$$\frac{(\lambda^2/2)\exp(-\lambda z_1)}{(\lambda_1\lambda_2/2)\exp[-(\lambda_2-\lambda_1)z_1/2-(\lambda_1+\lambda_2)z_1/2]} \leq K$$

for $z_2 - z_1$ yields a region of the form $\{(z_1, z_2): |z_2 - z_1| \leq g(k, z_1)\}$

called the critical region. Any observation falling in this region calls for rejecting the null hypothesis. The probability of an observation falling in the critical region is called the power of the test.

If time truncated testing had not been used and a trial solution of $|z_2| \geq (1-\alpha)z_1$ for the critical region is assumed, the power of the test is

$$\int_0^\infty \int_{(1-\alpha)z_1}^{z_1} f_{Z_1 Z_2} dz_2 dz_1 + \int_0^\infty \int_{-z_1}^{-(1-\alpha)z_1} f_{Z_1 Z_2} dz_2 dz_1$$

and if $\lambda_1 = \lambda_2 = \lambda$, this reduces to

$$\int_0^\infty 2\alpha\lambda^2 \exp[-\lambda z_1] dz_1 = \alpha.$$

When $\lambda_1 = \lambda_2$, the probability that an observation falls in the critical region is, with truncation,

$$\begin{aligned} & \int_0^T \int_{(1-\alpha)z_1}^{z_1} f_{Z_1 Z_2} dz_2 dz_1 + \int_0^T \int_{-z_1}^{-(1-\alpha)z_1} f_{Z_1 Z_2} dz_2 dz_1 \\ & + \int_T^{2T/(2-\alpha)} \int_{(1-\alpha)z_1}^{2T-z_1} f_{Z_1 Z_2} dz_2 dz_1 + \int_T^{2T/(2-\alpha)} \int_{z_1-2T}^{-(1-\alpha)z_1} f_{Z_1 Z_2} dz_2 dz_1 \\ & + \int_T^{2T/(2-\alpha)} \int_{z_2=2T-z_1} f_{Z_1 Z_2} dz_2 dz_1. \end{aligned}$$

This is 1 minus the value listed in Table 1.

The use of time truncation maps the region

$$A = \left\{ |z_2| \geq z_1 - \frac{2\alpha T}{2-\alpha}, z_1 \geq \frac{2T}{2-\alpha} \right\} \cup \left\{ |z_2| \geq (1-\alpha)z_1, T \leq z_1 \leq \frac{2T}{2-\alpha} \right\}$$

onto

$$B = \left\{ |z_2| = 2T - z_1, T \leq z_1 \leq \frac{2T}{2-\alpha} \right\} \cup \left\{ (1-\alpha)z_1 < |z_2| < 2T - z_1, T \leq z_1 \leq \frac{2T}{2-\alpha} \right\}$$

Clearly B is contained in A and A is contained in the critical region, thus under the null hypothesis the probability of observing $|z_2| \geq (1-\alpha)z_1$ is less than or equal to α . This is the probability of rejecting the null hypothesis when H_0 is true. Figure 1 displays a lower bound for the power of this test when $\alpha=.3$ and $T=21$.

If testing is not time truncated, a power function which is monotonically increasing in the absolute value of $(\lambda_2 - \lambda_1)Z_1$ is available. The test statistic is Z_2 given Z_1 , and the decision rule is: reject $H_0: \lambda_1 = \lambda_2$ if $Z_2 \geq (1-\alpha)Z_1$.

The power of the test in this case is

$$\begin{aligned} & 1 - \int_{-(1-\alpha)z_1}^{(1-\alpha)z_1} f_{Z_2|Z_1} dz_2 \\ &= 1 - \frac{\exp[(\lambda_2 - \lambda_1)(z_1 - \alpha z_1 - z_1)/2] - \exp[-(\lambda_2 - \lambda_1)(z_1 - \alpha z_1 + z_1)/2]}{1 - \exp[-(\lambda_2 - \lambda_1)z_1]} \\ &= 1 - \frac{\sinh [(\lambda_2 - \lambda_1)(1-\alpha)z_1/2]}{\sinh [(\lambda_2 - \lambda_1)z_1/2]} \end{aligned}$$

when $\lambda_1 = \lambda_2$ the power is α as may be verified by L' Hospital's rule.

Figure 2 displays the power function without time truncation when $\alpha=.1$ and for selected values of Z_1 .

The probability of type II error is the probability that an observation falls in the compliment of the critical region, $\{(z_1, z_2): |z_2| < (1-\alpha)z_1\}$, when H_0 is not true. This probability is

$$\begin{aligned}
 & \int_0^{2T/(2-\alpha)} \int_{-(1-\alpha)z_1}^{(1-\alpha)z_1} f_{Z_1|Z_2} dz_2 dz_1 \\
 & + \int_{2T/(2-\alpha)}^{\infty} \int_{\alpha 2T/(2-\alpha) - z_1}^{z_1 - (\alpha 2T)/(2-\alpha)} f_{Z_1|Z_2} dz_2 dz_1 \\
 (1.10) \quad & = 4(1-\alpha)(\lambda_1 \lambda_2) / \left[(2-\alpha)\lambda_1 + \alpha\lambda_2 \right] \left[(2-\alpha)\lambda_2 + \alpha\lambda_1 \right] \\
 & + \frac{\exp[-\lambda_1 T + \lambda_2 \alpha T / (2-\alpha)]}{(2-\alpha)\lambda_1 + \alpha\lambda_2} \lambda_2^\alpha + \frac{\exp[-\lambda_2 T + \lambda_1 \alpha T / (2-\alpha)]}{(2-\alpha)\lambda_2 + \alpha\lambda_1} \lambda_1^\alpha
 \end{aligned}$$

Table 1 shows this probability for selected values of λ_1, λ_2, T and α . When T becomes arbitrarily large, (1.10) reduces to $4(1-\alpha)(\lambda_1 \lambda_2) / \left[(2-\alpha)\lambda_1 + \alpha\lambda_2 \right] \cdot \left[(2-\alpha)\lambda_2 + \alpha\lambda_1 \right]$.

B. SUMMARY OF TEST PROCEDURE

The procedure for testing the hypothesis $H_0: \lambda_1 = \lambda_2$ against the hypothesis $H_A: \lambda_1 \neq \lambda_2$ with significance α

1. Observe Y_1 and Y_2
2. Form $Z_1 = Y_1 + Y_2$ and $Z_2 = Y_1 - Y_2$
3. Compute $(1-\alpha)Z_1$
4. If $Z_2 < (1-\alpha)Z_1$ accept H_0 .
If $Z_2 \geq (1-\alpha)Z_1$ reject H_0 .

Table I. Probability of Type II Error

λ_1	λ_2	T = 1		T = 6		T = 16	
		$\alpha = .1$	$\alpha = .3$	$\alpha = .1$	$\alpha = .3$	$\alpha = .1$	$\alpha = .3$
.001	.01	.9994	.998	.9965	.988	.9909	.969
	.1	.994	.982	.968	.899	.919	.754
	1	.948	.838	.730	.348	.434	.066
	10	.591	.171	.044	.00057	.0021	.00054
.01	.1	.994	.980	.967	.894	.921	.757
	1	.948	.837	.736	.361	.467	.099
	10	.592	.174	.057	.0055	.0187	.0054
.1	1	.947	.835	.787	.466	.679	.352
	10	.608	.198	.178	.052	.159	.052
1	10	.724	.384	.649	.344	.649	.344
10	100	.650	.344	.650	.344	.650	.344



$$\alpha = .3$$

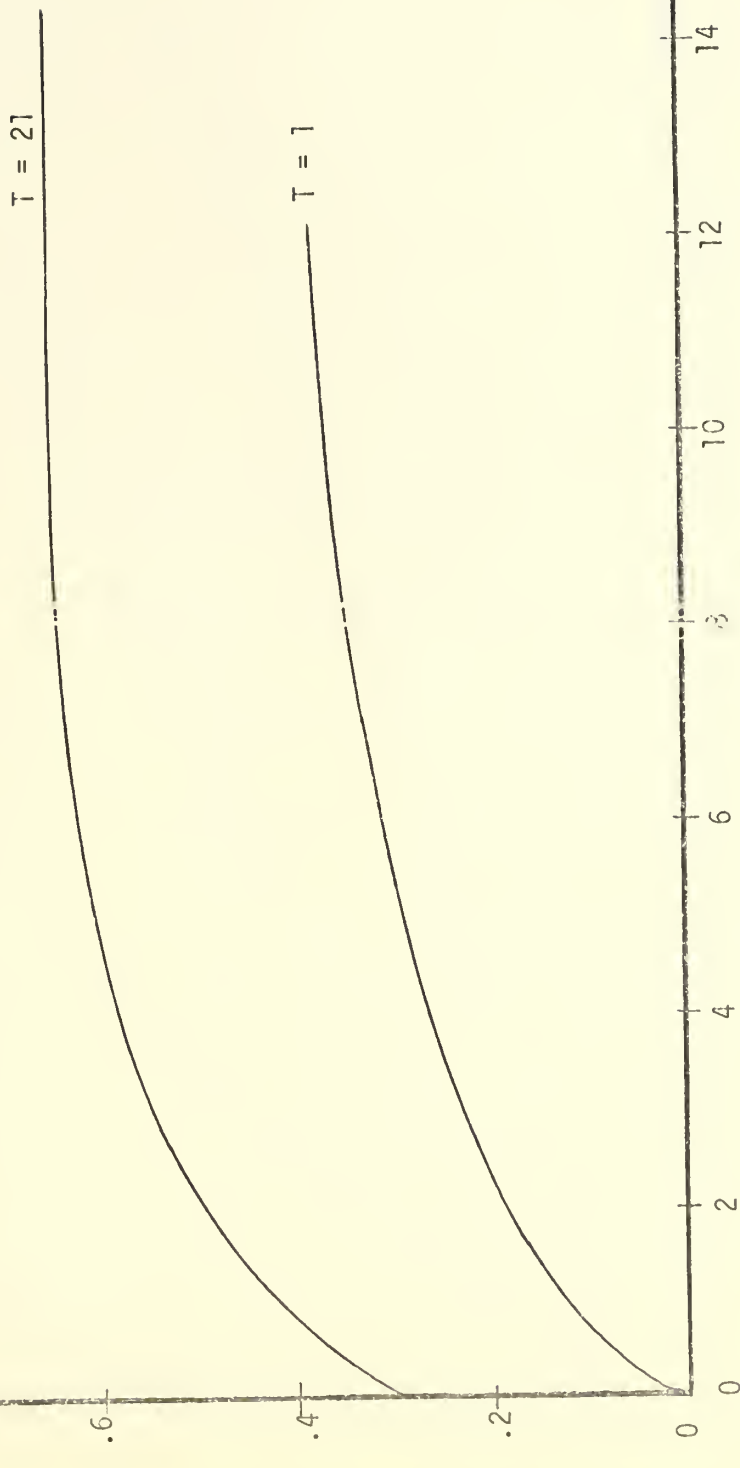


Figure 1. Power Function Lower Bound With Time Truncation.

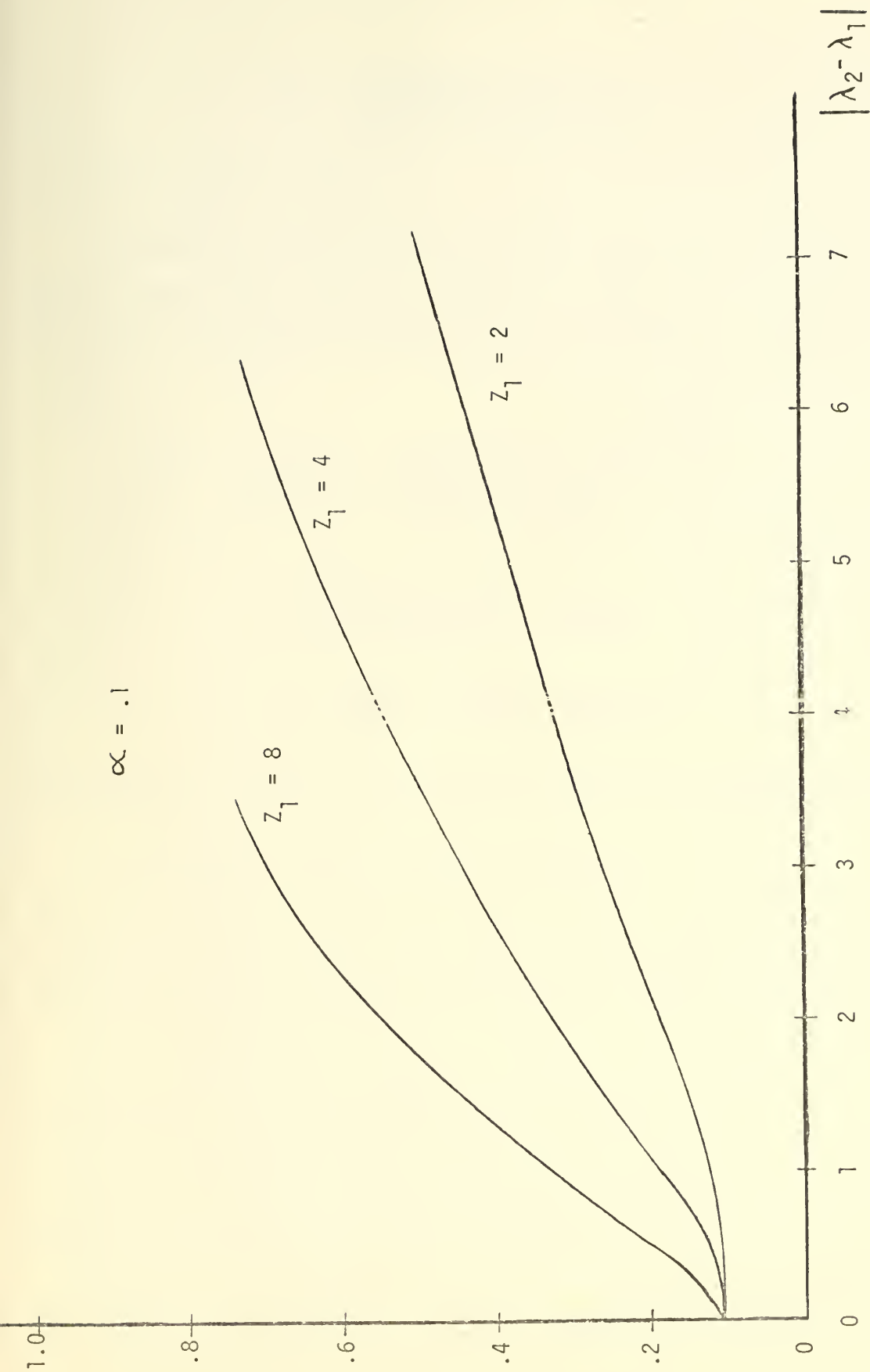


Figure 2. Power Function Without Time Truncation

III. ESTIMATION OF $\lambda_2 - \lambda_1$

A. POINT ESTIMATION OF $\lambda_2 - \lambda_1$

In section II, the density of Z_2 given Z_1 was derived for the case where neither X_1 nor X_2 are time truncated. This is the case $0 < Z_1 < T$ and the density is given by equation 1.8. The expected value of Z_2 given Z_1 is then

$$\begin{aligned} & \int_{-z_1}^{z_1} z_1 \left(\frac{\lambda_2 - \lambda_1}{2} \right) \frac{\exp[(\lambda_2 - \lambda_1)(z_2 - z_1)/2]}{1 - \exp[-(\lambda_2 - \lambda_1)z_1]} dz_2 \\ &= z_1 \frac{1 + \exp[-(\lambda_2 - \lambda_1)z_1]}{1 - \exp[-(\lambda_2 - \lambda_1)z_1]} - \frac{2}{\lambda_2 - \lambda_1} \\ &= z_1 \coth \left[(\lambda_2 - \lambda_1)z_1/2 \right] - 2/(\lambda_2 - \lambda_1). \end{aligned}$$

From this an estimate of the parameter $\lambda_2 - \lambda_1$ may be obtained by solving

$$z_2 = z_1 \coth \left[(\lambda_2 - \lambda_1)z_1/2 \right] - 2/(\lambda_2 - \lambda_1)$$

for $\lambda_2 - \lambda_1$. Numerical computations using Newton's method are appropriate with this equation. A typical example might be $z_1 = 2$, $z_2 = -1.32$. Then

$$-1.32 = 2 \coth \left[(\lambda_2 - \lambda_1) \right] - 2/(\lambda_2 - \lambda_1)$$

and the value of $\lambda_2 - \lambda_1$ which satisfies this is $\lambda_2 - \lambda_1 = -3$.

This result is also valid for the case $T \leq Z_1 < 2T$ and $Z_2 \neq 2T - Z_1$ since neither X_1 nor X_2 have been time truncated in this case.

For the case in which X_1 or X_2 are time truncated, a point estimate of $\lambda_2 - \lambda_1$ is not available. But a confidence interval for $\lambda_2 - \lambda_1$ may be obtained.

B. CONFIDENCE INTERVALS FOR $\lambda_2 - \lambda_1$

A $100(1-\alpha)\%$ confidence interval for a parameter is an interval on the real line, (a,b) , such that the estimator of that parameter has a $1-\alpha$ probability of being observed in the interval (a, b) and the unknown parameter will lie in the interval.

For the exponential random variable X with scale parameter λ , the unbiased estimator for λ is the observation X . A $100(1-\alpha)\%$ confidence interval for λ is

$$\left(\frac{-\ln(1-\alpha/2)}{X} , \frac{-\ln(\alpha/2)}{X} \right) .$$

To obtain a $(1-\alpha)$ 100% confidence interval for $\lambda_2 - \lambda_1$ when neither X_1 nor X_2 is time truncated, first obtain separately $(1-\delta)$ 100%, $\delta = 1 - \sqrt{1-\alpha}$, confidence intervals for λ_1 and λ_2 . Then by the transformation $\lambda = \lambda_2 - \lambda_1$, a confidence interval with confidence greater than or equal $(1-\alpha)$ 100% is

$$\left(-\frac{\ln \delta/2}{x_2} + \frac{\ln(1-\delta/2)}{x_1} , -\frac{\ln(1-\delta/2)}{x_2} + \frac{\ln \delta/2}{x_1} \right)$$

When either X_1 or X_2 are time truncated, a $(1-\alpha)$ 100% confidence interval for λ is

$$\left(0, -\frac{\ln \alpha}{T} \right)$$

again using the transformation $\lambda = \lambda_2 - \lambda_1$, assuming $Y_1 = y_1 < T$ and $Y_2 = T$, a confidence interval with confidence greater than or equal $(1-\alpha)$ 100% is

$$\left(\frac{\ln(1-\delta/2)}{y} , -\frac{\ln \delta}{T} + \frac{\ln \delta/2}{y} \right)$$

where $\delta = 1 - \sqrt{1-\alpha}$.

C. APPLICATIONS OF CONFIDENCE INTERVALS

An ad hoc procedure for accepting the hypothesis $\lambda_1 = \lambda_2$ based on confidence intervals for $\lambda_2 - \lambda_1$ would be to compute the confidence interval based on the observed values of the random variables Y_1 and Y_2 . Then if the point 0 is in the interval obtained, accept $H_0: \lambda_1 = \lambda_2$. This procedure corresponds favorably with the procedure proposed in Section II.

To illustrate this method and its correspondence with the proposed procedure of Section II, consider the observations

$$Y_1 = T = 1 \text{ and } Y_2 = .6666$$

Here $Z_1 = 1.6666$ and $Z_2 = .3333$. When $\alpha = .3$, $(1-\alpha)Z_1 = 1.1666$ which is greater than Z_2 , hence accept $H_0: \lambda_1 = \lambda_2$.

For $\delta = 1 - \sqrt{1-\alpha} = .165$, an 83.5% confidence interval for λ_1 is $(0, -\ln .165) = (0, 1.800)$ and an 83.5% confidence interval for λ_2 is

$$\begin{aligned} & ((-\ln .9175)/.666, (-\ln .0825)/.666) \\ & = (.129, 3.76) . \end{aligned}$$

Now, a confidence interval for $\lambda_2 - \lambda_1$ with confidence greater than or equal to 70% is

$$(-1.671, 3.76)$$

which includes the point zero, so the hypothesis $\lambda_1 = \lambda_2$ should be accepted. This agrees with the test result from Section II.

IV. APRIORI KNOWLEDGE OF λ

A. CHOICE OF TEST TIME T .

When the value of the scale parameter is known apriori, the experimenter may reduce the probability of type II error by a more advantageous choice of the test time T . If Y_2 is considered to be the standard, hence λ_2 known apriori, the probability of type II error, given by equation 1.10, can be bounded. Specifically, the terms involving $\lambda_2 T$ may be bounded by choosing T appropriately. For example, if the constraint

$$\exp[-\lambda_2 T] < \theta$$

is to be imposed, then T must be greater than $-\ln \theta / \lambda_2$.

For reliability studies, the scale parameters are usually known from previous data and hence can be of use for this type of test.

V. AREAS FOR FURTHER STUDY

A. EXTENSION OF RESULT TO SAMPLES GREATER THAN ONE

In order to extend the results obtained thus far to tests with more than one sample point from each population, it would be necessary to determine the joint density of the sum of the Z_{1i} random variables and the sum of the Z_{2i} random variables. Here the independence of random variables within populations and the independence between populations of the random variables makes the computations easy since

$$\widetilde{f}_{\sum_i Z_{1i}, \sum_i Z_{2i}}(s) = \prod_{i=1}^m \widetilde{f}_{Z_1, Z_2}(s)$$

where an equal number, m , of observations are taken from each population.

An intuitively appealing method of extending the result would be to test

$$\sum_i Z_{2i} \geq (1-\alpha) \sum_i Z_{1i}$$

and reject the hypothesis $H_0 = \lambda_1 = \lambda_2$ if the inequality was observed. Here one needs the density function of this test. Without truncation, the density of the sum $\sum_i Z_{2i}$ given the sum $\sum_i Z_{1i}$, is the density whose Laplace transform is

$$E \left[\exp \left(-s \sum_i Z_{2i} \right) \middle| \sum_i Z_{1i} \right] ,$$

which by independence is

$$\prod_{i=1}^m E \left[\exp \left(-s Z_{2i} \right) \middle| Z_{1i} \right] .$$

When $\lambda_1 = \lambda_2$, this is the produce of m Laplace transformations of uniform $(-z_{1i}, z_{1i})$ density functions, hence it is

$$\prod_{i=1}^m \left[\frac{\exp(sz_{1i}) - \exp(-sz_{1i})}{2sz_{1i}} \right]$$

$$= \frac{1}{s^m} \prod_{i=1}^m \left[\frac{\sinh(sz_{1i})}{z_{1i}} \right].$$

For m larger than 2, this is difficult to invert in order to obtain the density function.

Another possibility is to consider the index sets

$$J = \{i : Z_{1i} < T\}$$

$$K = \{i : T \leq Z_{1i} < 2T, Z_{2i} < 2T - Z_{1i}\}$$

$$L = \{i : Z_{2i} = Z_{1i} - T\}$$

$$M = \{i : Z_{2i} = Z_{1i} - 2T\}$$

$$N = \{i : Z_{1i} = 2T\}$$

and then consider a likelihood ratio test

$$\frac{f_{Z_{2i}, Z_{1i}}(z_{2i}, z_{1i} | \lambda_1 = \lambda_2)}{f_{Z_{2i}, Z_{1i}}(z_{2i}, z_{1i} | \lambda_1 \neq \lambda_2)} \leq K$$

Using the independence of the paired observations this ratio is the product over the respective index sets of the densities for the region corresponding to the index sets. Solving this for z_{2i} yields a critical region of the form

$$\sum_{i \in J} Z_{2i} + \sum_{i \in K} Z_{2i} + c_1 \left(\sum_{i \in L} Z_{2i} \right) + c_2 \left(\sum_{i \in M} Z_{2i} \right) > g(\{Z_{1i}\})$$

This suggests using the previously mentioned test

$$\sum_i z_{2i} \geq (1-\alpha) \sum_i z_{1i}$$

but with the modification of introducing scaling factors for pairs belonging to index sets L and M. Such a modification might be investigated as part of a future study.

VI. CONCLUSIONS

The test prescribed in this thesis has significance level less than or equal α , if time truncation is used, and significance level α , if time truncation is not used, when testing the equality of parameters for random variables with exponential distributions. As the parameter $\lambda_2 - \lambda_1$ moves away from the value zero, the power function increases.

When time truncation is not used, the power function is a monotonically increasing function of the absolute value of $(\lambda_2 - \lambda_1)Z_1$.

The test prescribed is applicable when testing with only a single sample from each population and it can be used instead of Cox's modified F test in the case of single samples when time truncation is used.

The use of confidence intervals described in this thesis is an extension of the procedure of Lentner and Buehler to the case of single samples from each population. This procedure may be used in the case of time truncated testing; the procedure of Lentner and Buehler may not. However, the confidence intervals used here are not exact, while those of Lentner and Buehler are.

BIBLIOGRAPHY

1. Lentner, M.M. and Buehler, R.J., "Some Inferences About Gamma Parameters with an Application to a Reliability Problem," Journal of the American Statistical Association, v. 58 p. 670-677, 1963.
2. Lloyd, D.K. and Lipow, M., Reliability, Prentice Hall, Inc., Englewood Cliffs, New Jersey, 1962.
3. Cox, D.R., "Some Applications of Exponential Ordered Scores," Journal of the Royal Statistical Society, Series B, v. 26, p. 103-110, 1964.
4. Gehan, R.A. and Thomas, D.G., "The Performance of Some Two-sample Tests in Small Samples With and Without Censoring," Biometrika, v. 56, p. 127-132, 1969.

INITIAL DISTRIBUTION LIST

	No. Copies
1. Defense Documentation Center Cameron Station Alexandria, Virginia 22314	2
2. Chief of Naval Personnel Pers - 11b Department of the Navy Washington, D.C. 20370	1
3. Library, Code 0212 Naval Postgraduate School Monterey, California 93940	2
4. Assoc Professor T. Jayachandran, Code 53JY Department of Mathematics Naval Postgraduate School Monterey, California 93940	1
5. Naval Postgraduate School Department of Operations Research and Administrative Sciences Monterey, California 93940	1
6. LT James N. Swan, USN 375c Bergin Monterey, California 93940	2

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

ORIGINATING ACTIVITY (Corporate author)

Naval Postgraduate School
Monterey, California 93940

2a. REPORT SECURITY CLASSIFICATION

Unclassified

2b. GROUP

REPORT TITLE

A Test of Two Unitary Exponential Samples With and Without Time Truncation

DESCRIPTIVE NOTES (Type of report and, inclusive dates)

Master's Thesis; September 1971

AUTHOR(S) (First name, middle initial, last name)

James Ned Swan

REPORT DATE

September 1971

7a. TOTAL NO. OF PAGES

30

7b. NO. OF REFS

4

3. CONTRACT OR GRANT NO.

9a. ORIGINATOR'S REPORT NUMBER(S)

5. PROJECT NO.

9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)

10. DISTRIBUTION STATEMENT

This document has been approved for public release and sale;
its distribution is unlimited.

11. SUPPLEMENTARY NOTES

12. SPONSORING MILITARY ACTIVITY

Naval Postgraduate School
Monterey, California 93940

13. ABSTRACT

A test of the equality of the parameters λ_1 and λ_2 of two exponential random variables X_1 and X_2 is developed and evaluated for the case of a single sample from each population. The test is valid with or without time truncated testing although the test is less powerful under time truncated testing. The results are applicable to reliability studies.

KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
nential Random Variables ability Studies idence Intervals						

Thesis

.S895 Swan

c.1 A test of two unitary
exponential samples with
and without time trun-
cation.

131517

BLINDERY
21000

Thesis

.S895 Swan

c.1 A test of two unitary
exponential samples with
and without time trun-
cation.

131517

thesS895

Test of two unitary exponential samples



3 2768 002 06011 3

DUDLEY KNOX LIBRARY